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AN ADAPTIVE AGE REPLACEMENT POLICY

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Abstract

Under an age replacement policy we replace at failure or at the end of a specified time interval, whichever occurs first. This makes sense if a failure replacement costs more than a planned replacement and the failure rate is strictly increasing. We assume that the failure distribution is a Weibull distribution with known shape parameter (greater than one) and unknown scale parameter. Moreover, we assume that a natural conjugate prior distribution with specified parameters is at hand which we modify after each stage according to Bayes' rule. Our policy adapts to the changing prior. We see intuitively that the larger the replacement interval set, the more information we are likely to obtain. We take account of this in a precisely defined manner via dynamic programming. The optimal policy is partially characterized and various limiting results are obtained.

Introduction and Summary

Under an age replacement policy, we replace at failure or at the end of a specified time interval, whichever occurs first. An age replacement policy makes sense when a failure replacement costs more than a planned replacement and the failure rate is striclty increasing.

The case where the failure distribution is known has been treated by several authors; see, e.g., [1] . When the criterion is expected cost per unit time over an infinite horizon, the optimal replacement interval to set is found as an elementary application of renewal theory. At the opposite extreme, we may know virtually nothing about the failure distribution. Moreover, as we acquire failure data, we may decide not to use the information. An intermediate case is where we have partial information about the failure distribution and adapt our replacement interval to additional information about the failure distribution acquired sequentially.

If our initial information about the failure distribution seems too diffuse to quantify sensibly, we must choose a policy that seems reasonable to use until enough data is accumulated. How much is enough seems to be a qualitative judgment. Scheduling no planned replacement for the first few stages is probably the best resort in view of the following result:

> With the criterion of expected cost per unit time over an infinite horizon, the minimax strategy over all failure distributions with the same mean is to replace only at failure.

For a proof, see [1, Section 4.2.1]. An analogous result for discounting is given in Appendix I.

The major assumption that we shall make is a strong one: viz., the failure distribution is known to be a Weibull distribution with given shape para-

meter greater than one; i.e.,

$$(1) \qquad F_\lambda(y) = \begin{cases} 1 - e^{-\lambda y^k}, & y \geq 0 \\ 0, \text{elsewhere} \end{cases}$$

$$k > 1 \text{ and known} .$$

This distribution has strictly increasing failure rate $\lambda k y^{k-1}$.

When λ is unknown, which except for Section I we assume to be the case, there is a trade-off between minimizing the expected loss on the current stage (= period starting just after the last replacement and ending just after the next replacement) and acquiring maximal information about the failure distribution so as to minimize future expected losses. An in-service failure is more informative than a planned replacement, because in the latter case all we would know is that the life of the item would have exceeded the replacement interval. The longer the replacement interval set, the more likely a failure replacement. However, setting an infinite replacement interval is clearly not necessarily optimal.

We take account of these intuitive considerations in a precisely defined way via the empirical Bayes approach [13] and dynamic programming.

Except for Section 8, we shall be dealing with an infinite horizon. Continuous discounting is used, with the loss incurred at the time of replacement and the total loss equal to the sum of the discounted losses incurred on the individual stages. Suppose that a stage starts at time t and we set a replacement interval a , chosen from the extended half line $[0, \infty]$. If a replacement actually occurs at $t+x$, then the loss incurred on that stage is

$$(2) \qquad L(a,x,t) = \begin{cases} c_1 e^{-\alpha(t+a)}, & \text{if } x = a \\ c_2 e^{-\alpha(t+x)}, & \text{if } x < a \end{cases}$$

where $0 < c_1 < c_2$ and α is a positive discount rate. The cost of a

planned (failure) replacement is $c_1(c_2)$.

Before going on to the main part of the paper (Sections 2-6), we consider in Section 1 the case where λ is known. In Section 2, we give the dynamic programming formulation of the (adaptive) case where λ is unknown. Various plausible limiting properties are shown to hold in Section 3. In Section 4, we show that an optimal policy exists. We consider in Section 5 an extension to the case where I (> 1) items are in operation simultaneously. In Section 6, we partially characterize the optimal policy and discuss its computation. In Section 7 (nonnegligible cost of adapting) and Section 8 (finite horizon), the loss function given by (2) is modified and a markedly different formulation results. This is followed by four appendices.

RELATED PROBLEMS. Adaptive maintenance policies for setting optimal inspection intervals were considered in [7]. The information-current loss trade-off is ignored, but taking account of it seems to be much harder for an inspection policy than for an age replacement policy. The difficulty is due to the apparently intractable likelihood function in the former case.

Consider, however, the following problem: determination of the optimal "burn-in" time (with respect to an appropriate loss function) to eliminate in-service failures due to "infant mortality". If the failure distribution is a Weibull distribution with known shape parameter $k < 1$, our methods can be readily applied. For a nonparametric approach to this problem based on the assumption of decreasing failure rate, see [10]. Note that failure times for those items that survive the "burn-in" period are not generally available.

1. Known Scale Parameters

In this section we assume that the scale parameter λ in (1) is known. Although this case is not our principal concern, the results, besides having intrinsic interest, will be needed later when we consider the asymptotic properties of an adaptive policy.

The expected loss due to replacement of an item put in service at time t is $e^{-\alpha t}\phi(a)$, where

$$\phi(a) = c_1 e^{-(\alpha a + \lambda a^k)} + c_2 \int_0^a e^{-\alpha x} dF_\lambda(x) . \tag{3}$$

From the fact that

$$\frac{\partial \phi(a)}{\partial a} = e^{-(\alpha a + \lambda a^k)} \left[(c_2 - c_1)\lambda k a^{k-1} - c_1 \alpha \right] , \tag{4}$$

we see that $\phi(a)$ looks like one of the following:

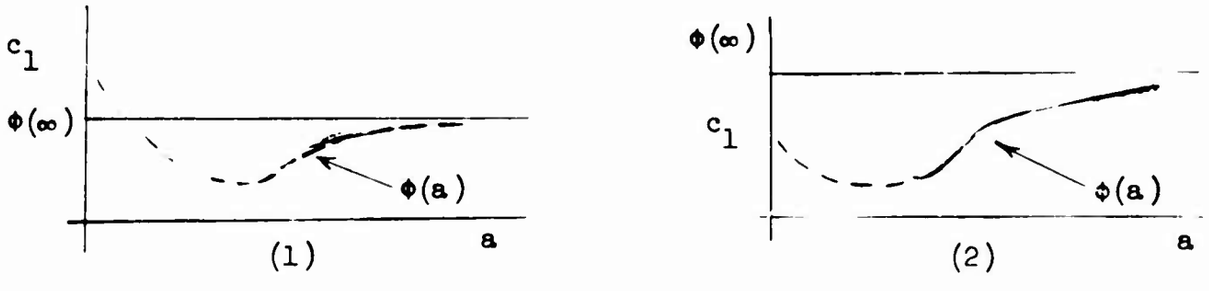

Figure 1. Possibilities for $\phi(a)$.

i.e., it crosses the line corresponding to no planned replacement once or not at all. As we shall see, the optimal a to set at each stage lies below the solid portion of the curve.

It is convenient to compute the risk starting at time 0. If the starting time is t, multiply by $e^{-\alpha t}$. Since the horizon is infinite, the (total) risk at each stage is the same, except for a discount factor; more-

over, the optimal replacement interval to set does not depend on the starting time. Due to this stationarity, there is some fixed replacement interval which is optimal to set at each stage.

Let X_i denote the age of the i^{th} item at its replacement time. When the same replacement interval is set each time, $\{X_i\}$ is a sequence of independent and identically distributed random variables. If we always choose a replacement interval a, the risk is

$$(5) \qquad R(a) = EL(a,X_1,0) + E\left[e^{-\alpha X_1} L(a,X_2,0)\right]$$

$$+ E\left[e^{-\alpha(X_1+X_2)} L(a,X_3,0)\right] + \cdots$$

$$= EL(a,X_1,0) + \left[Ee^{-\alpha X_1}\right]\cdot\left[EL(a,X_2,0)\right]$$

$$+ \left[Ee^{-\alpha X_1}\right]\cdot\left[Ee^{-\alpha X_2}\right]\cdot\left[EL(a,X_3,0)\right] + \cdots$$

$$= \phi(a) + \delta(a)\phi(a) + \delta^2(a)\phi(a) + \cdots$$

$$= \frac{\phi(a)}{1 - \delta(a)} \quad ,$$

where the expected discount per stage is

$$(6) \qquad \delta(a) = e^{-(\alpha a + \lambda a^k)} + \int_0^a e^{-\alpha x} dF_\lambda(x) \quad .$$

The relation $R(a) = \phi(a)/[1 - \delta(a)]$ can also be obtained directly from the optimality equation.

The minimum risk is

(7) $$R^* = \min_a R(a)$$

$$= R(a^*) \text{ , say .}$$

Immediate consequences of the theorem in Appendix II are

(i) $a^* < \infty$

(ii) a^* satisfies

(8) $$a^* = \left\{ \frac{c_1 + \Phi(a^*)/(1 - \delta(a^*))\alpha}{(c_2 - c_1)\lambda k} \right\}^{\frac{1}{k-1}}$$

(iii) a^* is unique. Although (8) may have more than one root, there is exactly one root of (8) that minimizes $R(a)$.

A procedure for finding successive approximations converging monotonely up to a^* that does not involve obtaining all the roots of (8) is given in Appendix II. From the first step of this procedure we have

(iv) $$a^* > \left[\frac{c_1 \alpha}{(c_2 - c_1)\lambda k} \right]^{\frac{1}{k-1}} ,$$

a result that can also be obtained from inspection of Fig. 1, the fact that $1 - \delta(a)$ is increasing, and a simple dominance argument.

2. Unknown Scale Parameters

Whereas the scale parameter was assumed known in Section 1, we shall assume from now on thatit has a fixed (but unknown) value λ^0. Moreover, we shall assume that we have at hand a prior distribution G with specified parameters which we modify after each stage according to Bayes' rule. If G has density g, then the posterior density = g x likelihood function x normalizing factor independent of λ. The likelihood function in case of planned replacement at a [failure replacement at x] is $e^{-\lambda a^k}$ $[k\lambda x^{k-1} e^{-\lambda x^k}]$. Taking

$$(9) \qquad g(\lambda;b,c) = \begin{cases} b^c \lambda^{c-1} e^{-b\lambda}/\Gamma(c), & \lambda \geq 0 \\ 0, \text{ elsewhere} \end{cases},$$

the posterior density is again a gamma density $g(\lambda;\ b+a^k,\ c)$ $[g(\lambda;\ b+x^k,\ c+1)]$. Thus, we see that we have a natural conjugate prior distribution [12] for λ. This makes a dynamic programming formulation possible with state variables (b,c,t), where (b,c) are the parameters of the prior distribution for a stage that begins at time t.

2.1 Preliminaries

Suppose that a replacement interval a is set and replacement occurs at t+x. Now we apply Bayes' rule: to obtain the new state variables, make the transformation

$$(10) \qquad (b,c,t) \longrightarrow \begin{cases} (b+a^k,\ c,\ t+a), & \text{if } x = a \\ (b+x^k,\ c+1,\ t+x), & \text{if } x < a \end{cases}.$$

Note that c increases in steps of 0 or 1, while b and t are restricted to half lines. Moreover, although the risk depends on t, the op-

timal policy does not. This means that the risk has to be computed only at points of a grid in R_2^+ ; other values are obtained by interpolation.

The reason for using a natural conjugate prior distribution is to keep the state space tractable, a point which was stressed in [3]. The restriction to the case where k is known is due to the apparent nonexistence of a natural conjugate prior distribution when k is unknown. Practical application of our results to cases where k is not precisely known depends on the sensitivity of the policy to the assumed value of k .

The expected loss from the next replacement when the state is (b,c,t) is $e^{-\alpha t}\,\phi(a,b,c)$, where (cf. (3) and (9))

$$(11)\qquad \phi(a,b,c) = c_1 e^{-\alpha a}\left(\frac{b}{b+a^k}\right)^c + c_2\, k\, c\, b^c \int_0^a \frac{e^{-\alpha x} x^{k-1}\,dx}{(b+x^k)^{c+1}} \quad .$$

For use throughout the sequel, it is convenient to define an operator T such that if $h(b,c)$ is an arbitrary function of (b,c), then

$$(12)\qquad T[a,h(b,c)] = \left(\frac{b}{b+a^k}\right)^c e^{-\alpha a} h(b+a^k,c) + kcb^c \int_0^a h(b+x^k,c+1)\,\frac{e^{-\alpha x} x^{k-1}\,dx}{(b+x^k)^{c+1}} \quad .$$

It is clear from (10) that we may interpret $T[a,h(b,c)]$ as the expected discounted value of $h(b',c')$, where (b',c',t') are the new state variables and a replacement interval a has been set.

2.2 Dynamic Programming Formulation

Following an adaptive policy indefinitely, the minimal risk $R(b,c)$ must

by the "principle of optimality" ([4],[8]) satisfy

$$R(b,c) = \min_{a} \{\Phi(a,b,c) + T[a,R(b,c)]\} . \tag{13}$$

No solution to this functional equation is apparent. However, we shall find two sequences $\{R_{1N}(b,c)\}$ and $\{W_{1N}(b,c)\}$ such that $R_{1N}(b,c)\downarrow R(b,c)$ and $W_{1N}(b,c)\uparrow R(b,c)$, provided in each case that the replacement intervals set are uniformly bounded away from 0. The proof of this statement for the first sequence is given in Section 3.2; for the second, the proof is similar.

We first find $R_{1N}(b,c)$. Instead of following an adaptive policy indefinitely, suppose that we were to follow an adaptive policy for N stages. From the $(N+1)$st stage onward, we would set the same replacement interval that we did on the N^{th} stage. Call the minimal risk at stage n of such a policy $R_{nN}(b,c)$. We obtain recursively

$$R_{NN}(b,c) = \min_{a} \tilde{\Phi}(a,b,c) \tag{14}$$

where

$$\tilde{\Phi}(a,b,c) = \frac{\Phi(a,b,c)}{1-\delta(a,b,c)} \tag{15}$$

and

$$\delta(a,b,c) = e^{-\alpha a}\left(\frac{b}{b+a^k}\right)^c + kcb^c \int_0^a \frac{e^{-\alpha x} x^{k-1} dx}{(b+x^k)^{c+1}} , \tag{16}$$

and for $n < N$

$$R_{nN}(b,c) = \min_{a} \{\Phi(a,b,c) + T[a,R_{n+1N}(b,c)]\} . \tag{17}$$

$W_{1N}(b,c)$ is found in a similar manner.

(18) $$W_{NN}(b,c) = \min_{a} \Phi(a,b,c)$$

and for $n < N$

(19) $$W_{nN}(b,c) = \min_{a} \{\Phi(a,b,c) + T[a,W_{n+1,N}(b,c)]\} \quad .$$

We may think of $W_{nN}(b,c)$ as the analog of $R_{nN}(b,c)$ for the case when we adapt for N stages and then terminate. It is felt that $\{R_{1N}(b,c)\}$ converges considerable faster than $\{W_{1N}(b,c)\}$.

$R_{nN}(b,c)$ is computed for a finite grid $\mathcal{G}$, $n = N,\ldots,1$. Starting from an initial value of c, the spacing between successive values of c is one (cf. (10)). For values of b off grid, $R_{nN}(b,c)$ is obtained by interpolation. We may choose to continue computing policies for successive values of N until

(20) $$\max_{(b,c)\in\mathcal{G}} [(R_{1,N-1}(b,c) - R_{1N}(b,c))/R_{1,N-1}(b,c)] < \epsilon \quad ,$$

where $\epsilon > 0$ is a preassigned tolerance. A more stringent criterion would be to continue computing until

(21) $$\max_{(b,c)\in\mathcal{G}} [(R_{1N}(b,c) - W_{1N}(b,c))/R_{1N}(b,c)] < \epsilon \quad .$$

Observe that, if (21) is satisfied, it still holds with $W_{1N}(b,c)$ replaced by $R(b,c)$. Although we would use (21) with more confidence than (20), much more computation would be involved.

Let $a_{nN}(b,c)$ be the minimizing a in (17) [in (14), if $n = N$]. Always acting as if we were at the first stage, what we shall actually do at stage n ($n = 1,2,\ldots,N^*,\ldots$) is to use the replacement interval $a_{1N^*}(b_n,c_n)$, where the state variables at stage n are (b_n,c_n,t_n) and N^* is the smallest N satisfying (20) or (21), whichever is deemed appropriate.

3. Limiting Properties

Analogously to $R_{nN}(b,c)[R(b,c)]$, define $R^{\epsilon}_{nN}(b,c)[R^{\epsilon}(b,c)]$ to be the minimal risk under the restriction that the replacement interval set at each stage be at least ϵ, for some fixed $\epsilon > 0$. Let $a^{\epsilon}_{nN}(b,c)[a^{\epsilon}(b,c)]$ denote the minimizing a. Our replacement policy will use $a^{\epsilon}_{1N}(\cdot,\cdot)$ at each stage. We call this policy (N,ϵ)-optimal.

In applications, restriction to the class of (N,ϵ)-optimal policies is no important. It is plausible that our results hold without the restriction that the replacement intervals set are uniformly bounded away from 0, but proofs are lacking.

We may paraphrase (roughly) the main results of this chapter as follows:

1. After a sufficiently large number of replacements, we are virtually certain at any particular stage to be setting a replacement interval close to the one we would use if we knew λ^0.
2. For N sufficiently large, $a^{\epsilon}_{1N}(\cdot,\cdot)$ is "close" to $a^{\epsilon}(\cdot,\cdot)$.

We also consider what happens for small discount rate α.

3.1 Asymptotic Optimality

An (N,ϵ)-optimal policy seems to be a reasonable one to use during the transient period when we are acquiring significant information about the failure distribution. Because of discounting, this is an important consideration. Moreover, we now state a result that shows that an (N,ϵ)-optimal policy has an asymptotic optimality property that we would require of any policy we were to use.

THEOREM 1. (asymptotic optimality). Under an (N,ϵ)-optimal policy,

$$\operatorname*{plim}_{\substack{n\to\infty \\ N \text{ fixed}}} R_{\lambda 0}\left[a^{\epsilon}_{1N}(b_n,c_n)\right] = R^{*}_{\lambda 0} \quad ,$$

provided that $a^{*}_{\lambda 0} \geq \epsilon$. In fact, with the same proviso,

$$\text{plim } a^{\epsilon}_{1N}(b_n,c_n) = a^{*}_{\lambda 0} \quad .$$

To prove this theorem, we require some preliminary results. To save tedious repetition, we do not mention Slutsky's theorem [5, Section 20.6] each time we use it.

LEMMA 1. Let Y have distribution F_λ and $X = \min(a,Y)$, where a is a positive constant. The variance of X^k is maximized at $a = \infty$ and is finite.

PROOF.

$$EX^k = \frac{1}{\lambda}\left(1 - e^{-\lambda a^k}\right)$$

$$\text{Var } X^k = \frac{1 - e^{-2\lambda a^k}}{\lambda^2} - \frac{2a^k e^{-\lambda a^k}}{\lambda} \quad .$$

$$\frac{\partial}{\partial a} \text{Var } X^k = 2ka^{2k-1}e^{-\lambda a^k} > 0 \quad . \quad \|$$

LEMMA 2. Under any policy for which a_n, the n^{th} replacement interval set, must be at least $\epsilon > 0$, $n = 1,2,\ldots,$

(i) $$\text{plim } b_n/n = \xi/\lambda^0$$

(ii) $$\text{plim } c_n/n = \xi \quad ,$$

where ξ is a positive constant.

PROOF. Let $X_n = \min(a_n, Y)$, where Y has distribution F_{λ^0}. By Lemma 1, the variance of X_n^k is uniformly bounded over n for all $(a_1, a_2, \ldots)$; the bound is achieved for $(\infty, \infty, \ldots)$.

Therefore, by a standard martingale convergence theorem [11, Section 29.1],

$$\text{plim } \frac{b_n}{n} = \text{plim } \frac{1}{n}\left(b_1 + \sum_1^n X_1^k\right)$$

$$= \text{plim } \frac{1}{n}\sum_1^n X_i^k$$

$$= \text{plim } \frac{1}{n}\sum_1^n E(X_1^k \mid X_1, \ldots, X_{i-1})$$

$$= \frac{1}{\lambda^0}\,\text{plim } \frac{1}{n}\sum_1^n (1 - e^{-\lambda^0 a_1^k}) \quad .$$

The limit is a positive constant, since $a_n \geq \epsilon > 0$, $\forall_n$ and $\text{Var } b_n/n \to 0$.

Hence, (i) is proved. The proof of (ii) is similar. ||

LEMMA 3. (consistency). Under any policy for which $a_n \geq \epsilon > 0$, $\forall_n$, $P[G(\lambda; b_n, c_n)$ converges to a distribution degenerate at $\lambda^0] = 1$.

PROOF. The mean and variance of the prior distribution at stage n are, respectively, c_n/b_n and c_n/b_n^2. By Lemma 2

$$\text{plim } \frac{c_n}{b_n} = \text{plim } \frac{c_n/n}{b_n/n}$$

$$= \frac{\text{plim } c_n/n}{\text{plim } b_n/n} = \lambda^0 \ ;$$

similarly,

$$\text{plim } \frac{c_n}{b_n^2} = \text{plim } \frac{c_n}{b_n} \quad \text{plim } \frac{1}{b_n}$$

$$= \lambda^0 \ \text{plim } \frac{1}{b_n} = 0 \ ,$$

since plim $b_n = \infty$ by Lemma 2. Since convergence in quadratic mean $\Longrightarrow$ convergence in probability $\Longrightarrow$ convergence of laws, the lemma follows. ||

PROOF OF THEOREM 1. We observe that Lemma 2 implies

$$\text{plim } b_n = \infty$$

$$\text{plim } c_n = \infty$$

and hence it follows from inspection of (11), (12), and (14)-(16) that with b_n and c_n random variables and $n \to \infty$

$$T\left[a, R^{\epsilon}_{NN}(b_n, c_n)\right] = \delta(a, b_n, c_n) R^{\epsilon}_{NN}(b_n, c_n) + o(1) \tag{22}$$

w.p.1

Next, we note that (14) and (15) imply

$$R^{\epsilon}_{NN}(b,c) = \min_{a \geq \epsilon} \left[\phi(a,b,c) + \delta(a,b,c) R^{\epsilon}_{NN}(b,c)\right] \ . \tag{23}$$

Hence

$$(24)\qquad R^{\epsilon}_{N-1,N}(b_n,c_n)$$

$$= \min_{a \succeq \epsilon} \left\{ \Phi(a,b_n,c_n) + \delta(a,b_n,c_n)\left[R^{\epsilon}_{NN}(b_n,c_n) + o(1) \right] \right\}$$

w.p.1 by (17) and (22)

$$= R^{\epsilon}_{NN}(b_n,c_n) + o(1)$$

w.p.1 by (23) (comparing minimands)

$$= \min_{a \succeq \epsilon} \left\{ \Phi(a,b_n,c_n) + \delta(a,b_n,c_n)\left[R^{\epsilon}_{N-1,N}(b_n,c_n) + o(1) \right] \right\}$$

w.p.1

Letting $\tilde{a}_n$ stand for $a^{\epsilon}_{1N}(b_n,c_n)$, we have, by backwards induction (on the hypothesis $R^{\epsilon}_{nN}(b_n,c_n) = R^{\epsilon}_{NN}(b_n,c_n) + o(1)$ w.p.1, $n = N-1,\ldots,n'$)

$$(25)\qquad , \quad R^{\epsilon}_{1N}(b_n,c_n) = \Phi(\tilde{a}_n,b_n,c_n) + \delta(\tilde{a}_n,b_n,c_n)\left[R^{\epsilon}_{1N}(b_n,c_n) + o(1) \right]$$

w.p.1; hence

$$(26)\qquad R^{\epsilon}_{1N}(b_n,c_n) = \frac{\Phi(\tilde{a}_n,b_n,c_n) + o(1)}{1 - (\tilde{a}_n,b_n,c_n)} \qquad \text{w.p.1,}$$

and so

$$(27)\qquad \operatorname{plim} \; R^{\epsilon}_{1N}(b_n,c_n)$$

$$= \operatorname{plim} \min_{a \succeq \epsilon} \left[\frac{\Phi(a,b_n,c_n) + o(1)}{1 - \delta(a,b_n,c_n)} \right] \qquad .$$

We note that by the Hally-Bray theorem [11, Section 11.3] and Lemma 3

$$\text{plim } \Phi(a,b_n,c_n) = \Phi_{\lambda 0}(a) \tag{28}$$

$$\text{plim } \delta(a,b_n,c_n) = \delta_{\lambda 0}(a) \quad . \tag{29}$$

Since the minimand in (27) converges uniformly for $a \geq \epsilon$, we may interchange plim and min, which is justified by the lemma proved in Appendix III. Therefore, by (28) and (29)

$$\text{plim } R^{\epsilon}_{1N}(b_n,c_n) = \min_{a \geq \epsilon} \left[\frac{\Phi_{\lambda 0}(a)}{1 - \delta_{\lambda 0}(a)} \right] \tag{30}$$

$$= R^{*}_{\lambda 0} \quad , \quad \text{if} \quad a^{*}_{\lambda 0} \geq \epsilon \quad .$$

Let

$$\Delta_n(a) = \left| \frac{\Phi_{\lambda 0}(a)}{1 - \delta_{\lambda 0}(a)} - \frac{\Phi(a,b_n,c_n)}{1 - \delta(a,b_n,c_n)} \right| \tag{31}$$

Since by (28) and (29) $\Delta_n(a)$ converges in probability to 0 uniformly for $a \geq \epsilon$, $\text{plim } \Delta_n(\tilde{a}_n) \leq \text{plim } \sup_{a \geq \epsilon} \Delta_n(a) = \sup_{a \geq \epsilon} \text{plim } \Delta_n(a) = 0$ and hence

$$\text{plim } \left| R_{\lambda 0}(\tilde{a}_n) - R^{\epsilon}_{1N}(b_n,c_n) \right| = 0 \quad . \tag{32}$$

Combining (32) with (30) shows that

$$\text{plim } R_{\lambda 0}(\tilde{a}_n) = R^{*}_{\lambda 0} \quad , \quad \text{if} \quad a^{*}_{\lambda 0} \geq \epsilon \quad , \tag{33}$$

proving the first assertion of the theorem. Since by the theorem of Appendix

II $a^*_{\lambda 0}$ is unique, the second assertion follows from the first. ||

3.2 Behavior as $N \to \infty$

Let $R(\pi,b,c)$ denote the risk under the stationary policy π. For example, if we use $a^{\epsilon}_{1N}(\cdot,\cdot)$ at each stage, then $\pi = a^{\epsilon}_{1N}(\cdot,\cdot)$. Note that $R(a^{\epsilon}(\cdot,\cdot),b,c) = R^{\epsilon}(b,c)$. If we measure the "distance' from π^1 to π^2 by

$$|| \pi^1 - \pi^2 || = \sup_{(b,c)\in\Gamma} | R(\pi^1,b,c) - R(\pi^2,b,c) | ,$$

where Γ is a compact subset of $\{(b,c): b > 0, c > 0\}$, then we have

THEOREM 2. $\lim_{N \to \infty} || a^{\epsilon}_{1N}(.,.) - a^{\epsilon}(.,.) || = 0$.

Two lemmas are required for the proof.

LEMMA 4. $R^{\epsilon}(b,c)$ satisfies

$$R^{\epsilon}(b,c) = \min_{a \geq \epsilon} \{\Phi(a,b,c) + T[a,R^{\epsilon}(b,c)]\} \quad ; \tag{34}$$

moreover, if $H(b,c)$ is a solution to (34) such that

$$\text{plim } e^{-\alpha t_n} H(b_n,c_n) = 0 \quad , \tag{35}$$

then $H(b,c) = R^{\epsilon}(b,c)$.

A proof can be given along the lines of [8].

LEMMA 5.

(i) $R^{\epsilon}_{1N}(b,c) \downarrow R^{\epsilon}(b,c)$.

(ii) $T[\cdot, R^{\epsilon}(b,c)]$ is continuous .

(iii) convergence is uniform on Γ .

(iv) $R^{\epsilon}(\cdot,\cdot)$ is continuous on Γ .

PROOF. Since $R^{\epsilon}_{1N}(b,c)$ is bounded below by 0 and decreasing in N , $R^{\epsilon}_{1N}(b,c) \downarrow H(b,c)$, say. Since Theorem 1 implies that plim $e^{\alpha t_n} R^{\epsilon}_{11}(b_n,c_n) = 0$, it follows that (35) holds. By Lemma 4, it therefore suffices to show that $H(b,c)$ satisfies (34) to prove (i) .

Taking the limit as $N \to \infty$ on both sides of

$$R^{\epsilon}_{1N}(b,c) = \min_{a \geq \epsilon} \{\Phi(a,b,c) + T[a, R^{\epsilon}_{2N}(b,c)]\} ,$$

we get $H(b,c)$ on the left. On the right

(I) interchange lim and min

This is justified by the lemma proved in Appendix III.

(II) take the limit inside the integral in the relation

$$T[a, R^{\epsilon}_{2N}(b,c)] = T[a, R^{\epsilon}_{1,N-1}(b,c)]$$

$$= \left(\frac{b}{b+a^k}\right)^c e^{-\alpha a} R^{\epsilon}_{1,N-1}(b+a^k, c)$$

$$+ kcb^c \int_0^a R^{\epsilon}_{1,N-1}(b+x^k, c+1) \frac{e^{-\alpha x} x^{k-1} dx}{(b+x^k)^{c+1}} .$$

This is justified by the Lebesgue monotone convergence theorem.

This shows that $H(b,c)$ satisfies (34).

It is easily seen that (i) $\longrightarrow$ (iii). Since the uniform limit of continuous functions is continuous, (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv). ‖

Theorem 2 is an immediate consequence of part (iii) of Lemma 5. Note that the convergence of $\| a^{\epsilon}_{1N}(\cdot,\cdot) - a^{\epsilon} \|$ to 0 is not necessarily monotone.

Provided that the unconstrained minimizers are $\geq \epsilon$, we conjecture that, starting with $N = 2$, $a^{\epsilon}_{1N}(b,c) \downarrow a^{\epsilon}(b,c)$ but that $a^{\epsilon}_{11}(b,c) \leq a^{\epsilon}_{1N}(b,c)$, $N = 2,3,\ldots$. The intuitive basis for these conjectures is that

(i) the longer the replacement interval set, the more information about the failure distribution we are likely to obtain;

(ii) the larger N, the more stages we have to get information and so acquiring maximal information becomes relatively less important;

(iii) an exceptional case occurs for $N = 1$ because there we do not take account of information acquisition at all.

In contrast to these conjectures, any of the possibilities

$$a^{\epsilon}_{1N}(b_n,c_n) \lesseqgtr a^{\epsilon}_{1N}(b_{n+1},c_{n+1})$$

can hold.

3.3 Behavior as $\alpha \to 0$.

To emphasize the dependence of $\tilde{\Phi}(a,b,c)$ on α, we write $\tilde{\Phi}(a,b,c;\alpha)$. From a well-known Tauberian theorem [14, Chapter 5]

on Laplace-Stieltjes transorms, it follows that the expected cost per unit time (with respect to our current prior distribution) when the same replacement interval a is set at each stage is

$$\lim_{a \to 0^+} \alpha\tilde{\Phi}(a,b,c;\alpha) \quad .$$

Thus, by taking α small we could approximate what we would do if our goal were actually to minimize expected cost per unit time.

For a discussion of this point in a somewhat different context, see [6].

Note that the (adaptive) problem degenerates when the loss is undiscounted cost per unit time since we could ignore the loss in any finite transient period while we learned about the failure distribution.

4. Existence of an Optimal Policy

For any particular (b,c), we have already seen (cf. Lemma 4) that an optimal replacement interval to set exists, viz. $a^{\epsilon}(b,c)$. Although we do not have a means for obtaining it exactly, by successive approximations we can do almost as well (cf. Theorem 2). From a practical viewpoint, perhaps this is all we should ask.

However, taking a global outlook, a policy $\pi = (\pi_1,\pi_2,\dots)$ is a sequence of functions such that π_n tells us what action to take as a function of (b_n,c_n). (We lose nothing here by considering only nonrandomized policies.) In view of the functional equation (34), we always act as if we were at the first stage. Hence, our policy is stationary, i.e., $\pi = (\pi_1,\pi_1,\dots)$. Following [8], it can be shown that an optimal policy exists with respect to the topology of pointwise convergence [9, Chapter 7] using Tychonoff's theorem on the prodict of compact spaces. Of course, the optimal policy is to use $a^{\epsilon}(\cdot,\cdot)$ at each stage.

5. Extension

Until now we have been tacitly assuming that there was only one item in operation at any particular time. However, suppose that I items are in operation simultaneously. Our previous results can be applied to this case. For simplicity, we consider only the case $I = 2$.

Without loss of generality, we assume that just after the last replacement item 1 had age A, item 2 was new, and the state was (b,c,t). Suppose that we are using an "N" stage policy. Let $a = a_{1N}(b,c)$. We set replacement intervals $\max(0,a-A)$ and a for items 1 and 2, respectively.

Case 1: Neither item fails before $\max(0,a-A)$.

(i) $(b,c,t) \rightarrow (b+a^k + \max(0,a^k - A^k,c,t+\max(0,a-A))$.

(ii) replace the older item if $A > 0$; otherwise both items.

(iii) in the former case, the item not replaced has age $\max(0,a-A)$.

(iv) proceed as before.

Case 2: The older (newer) item is the first to fail; it fails at $t + x$.

(i) $(b,c,t) \rightarrow (b+2x^k-A^k,c+1,t+x)$.

(ii) replace the older (newer) item.

(iii) the item not replaced has age x $(A+x)$.

(iv) proceed as before.

Note that we will often modify a replacement interval previously set.

6. Computing the Optimal Policy

Section 6.1 is preliminary. In Section 6.2 we partially characterize the optimal policy. This characterization can be used to expedite the search for minimizers in (14) and (17).

In computing the optimal policy, errors can arise in two ways:

(i) in numerical integration and in the search for minimizers, we must in practice confine ourselves to a finite interval.

(ii) because we are restricted to a finite grid where $R_{nN}(b,c)$ is computed recursively for $n = N,\ldots,1$, there will be errors due to interpolation as well as numerical integration.

We bound the error corresponding to (i) in Section 6.3. The magnitude of the error corresponding to (ii) depends on the fineness of the grid. In any case, the spacing between successive values of c is one. However, there is a tradeoff in the spacing of b values between error reduction and computing time. No doubt it is efficient to use unequal spacings determined during the course of the computation, but we shall not consider the matter further.

An outline of the replacement policy computation is given in algorithmic form in Appendix V.

6.1 Possibilities for the Graph of $\Phi(a,b,c)$

In contrast to the situation where λ is known, $\Phi(a,b,c)$ looks like one of the five possibilities shown in Fig. 2. Their significance will be brought out in Theorem 3.

To see that one of these cases must hold, we note that the slope of $\Phi(a,b,c)$, given by

$$(36)\qquad \frac{\partial}{\partial a}\Phi(a,b,c) = e^{-\alpha a}\left(\frac{b}{b+a^k}\right)^c \left[\frac{kc(c_2-c_1)a^{k-1}}{b+a^k} - c_1\alpha\right],$$

is negative at $a = 0^+$ and for all sufficiently large a. The case that occurs depends on the number of positive zeros of $\frac{\partial}{\partial a}\Phi(a,b,c)$. The correspondence is

no. positive zeros	case
2	(1) - (3)
0	(4)
1	(5)

The number of positive zeros is 2, 0, 1 as

$$(37)\qquad \alpha \begin{matrix} < \\ > \\ = \end{matrix} \max_{a} \frac{kc(c_2-c_1)a^{k-1}}{c_1(b+a^k)} .$$

By setting the derivative of the maximand equal to zero, we find the maximum occurs at $a = [(k-1)b]^{1/k}$ and that (37) is equivalent to

$$(38)\qquad b \begin{matrix} < \\ > \\ = \end{matrix} [c(c_2-c_1)/c_1\alpha]^k (k-1)^{k-1}$$

Since the derivative of the maximand has exactly one zero,

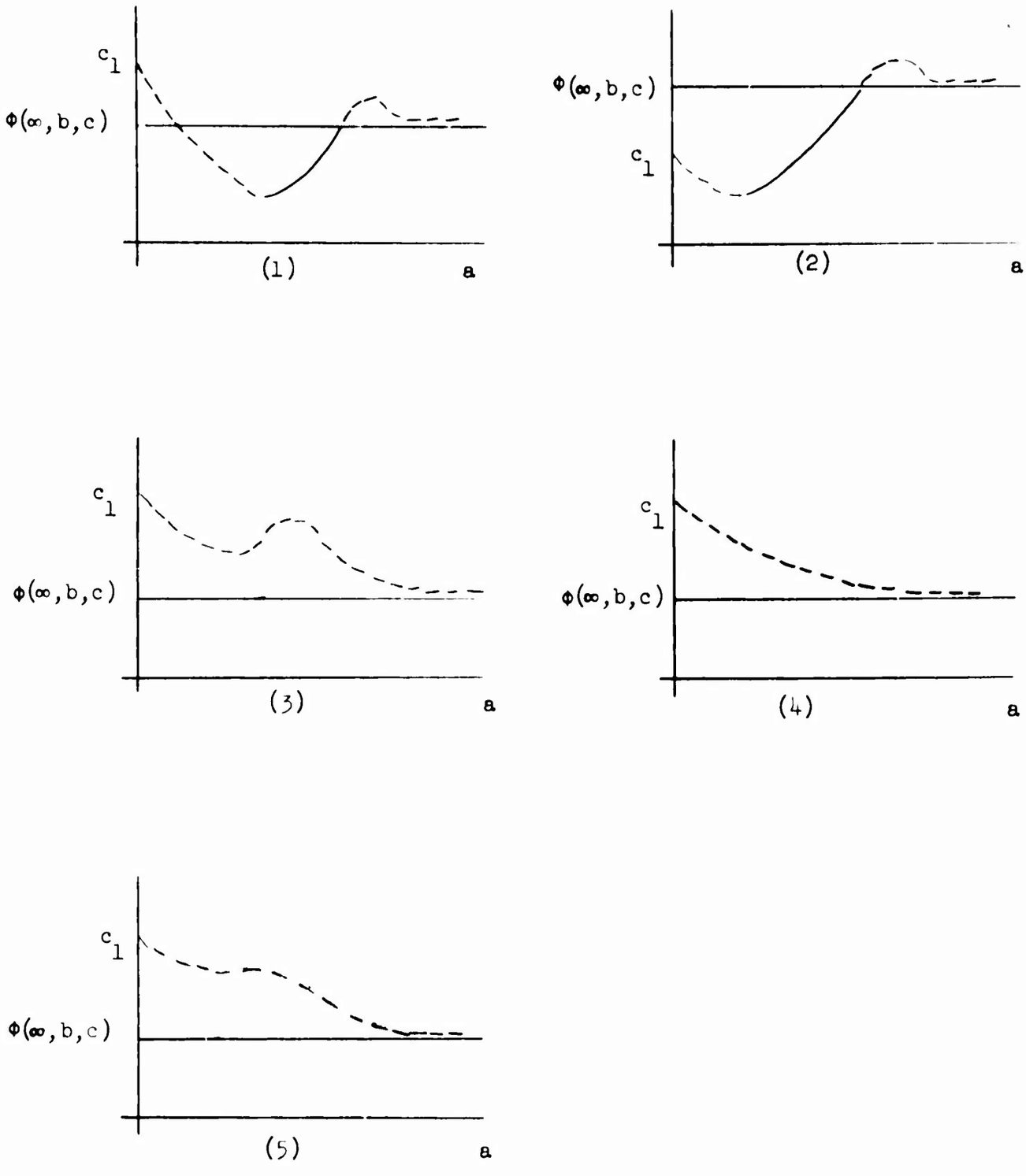

Figure 2. Possibilities for $\Phi(a,b,c)$.

$\frac{\partial}{\partial a}\,\Phi(a,b,c)$ can cross the abscissa at most twice; i.e., $\frac{\partial}{\partial a}\,\Phi(a,b,c)$ has at most two positive zeros.

We remark that for α sufficiently small case (2) must hold.

6.2 Partial Characterization of the Optimal Policy

THEOREM 3. For $n = N$:

(i) $b \geq [c(c_2-c_1)/c_1\alpha]^k (k-1)^{k-1} \qquad a_{nN}(b,c) = \infty$.

(ii) Suppose that the inequality in (i) does not hold. Let $a_S(b,c)$ denote the smaller positive zero of $\frac{\partial}{\partial a}\,\Phi(a,b,c)$. Then

$$\Phi(a_S,b,c) \geq \Phi(\infty,b,c) \qquad a_{nN}(b,c) = \infty \quad .$$

(iii) Suppose that neither of the preceding inequalities holds. Let $a_L(b,c)$ denote the root of $\Phi(a,b,c) = \Phi(\infty,b,c)$ that is larger than a_S. Then either $a_{nN}(b,c) = \infty$ or $a_S(b,c) < a_{nN}(b,c) < a_L(b,c)$.

PROOF. Using simple dominance arguments, the theorem follows from the fact that $1-\delta(a,b,c)$ is increasing in a, and inspection of Figure 2. Note that case (4) or case (5) holds iff the inequality in (i) holds. ‖

We conjecture that

(I) $$\frac{\partial}{\partial a} T[a,R_{nN}(b,c)] < 0 , \quad n = N,\ldots,1$$

(II) The extension of Theorem 3 to $n = N,\ldots,1$ holds.

Note that (I) $\Longrightarrow$ (II).

6.3 An Error Bound

Let $0 < \epsilon < \min(1,c_1)$,

(39) $$\nu_1 = [c(c_2-c_1)/c_1\alpha]^k (k-1)^{k-1}$$

(40) $$\nu_2 = 2[(c_2+1)(c+1)\alpha^{-k}\Gamma(k+1)]/\epsilon ,$$

and ν_3 be the smallest a satisfying

(41) $$c_1 e^{-\alpha a}\left(\frac{b}{b+a^k}\right)^c \leq \epsilon/2 .$$

We shall bound the error introduced by the following

PROCEDURE. Select a number a', depending on (b,c), such that

(42) $$a' = \max\left[(\max(\nu_1,\nu_2) - b)^{1/k}, \nu_3\right] .$$

Take the minimizer over $[0,a']$ to be the replacement interval, unless the minimizer is a'; in the later case, take the replacement interval

to be ∞, approximating $R_{nN}(b,c)$ by

$$(43)\qquad \begin{cases} \tilde{\phi}(a',b,c), & \text{if } n = N \\ \phi(a',b,c) + T[a', R_{n+1,N}(b,c)], & \text{if } n < N . \end{cases}$$

THEOREM 4. The risk under the foregoing procedure exceeds the minimal risk over all procedures using the grid $\mathfrak{b}$ by at most

$$2\epsilon(N-1) + \frac{\epsilon}{1 - M} ,$$

where

$$(44)\qquad M = \max(M', M'')$$

$$(45)\qquad M' = \max\{\delta(a',b,c) : (b,c)\epsilon\mathfrak{b},\quad b \geq \nu_1(c)\}$$

$$(46)\qquad M'' = \max\{\delta(a_S(b,c),b,c) : (b,c)\epsilon\mathfrak{b},\quad b < \nu_1(c)\}$$

and $a_S(b,c)$ is the smaller positive zero of $\frac{\partial}{\partial a}\phi(a,b,c)$.

To prove Theorem 4, we need two lemmas.

LEMMA 6. If

$$(47)\qquad b > \max\left[\nu_1, (c+1)\alpha^{-k}\Gamma(k+1)\right] ,$$

then, for $n = N,\ldots,1$,

$$(48)\qquad R_{nN}(b+a^k,c) \leq \frac{c_2 c\alpha^{-k}\Gamma(k+1)/(b+a^k)}{1-c\alpha^{-k}\Gamma(k+1)/(b+a^k)}$$

and

$$(49)\qquad kcb^c \int_{a'}^{a} R_{nN}(b+x^k,c+1)\,\frac{e^{-\alpha x}x^{k-1}dx}{(b+x^k)^{c+1}} \leq$$

$$\frac{c_2[(c+1)\alpha^{-k}\Gamma(k+1)/(b+a'^k)]^2}{1-(c+1)\alpha^{-k}\Gamma(k+1)/(b+a'^k)} \quad .$$

PROOF. Observe that (47) implies that the denominators on the right-hand sides of (48) and (49) are positive.

It suffices to show that (48) and (49) hold when $n = N$, since $R_{nN}(b,c) \leq R_{NN}(b,c)$. By Theorem 3(i), (47) implies

$$a_{NN}(b,c) = \infty \quad . \tag{50}$$

Note that

$$\phi(\infty,b,c) = c_2 k c b^c \int_0^\infty \frac{e^{-\alpha x} x^{k-1} dx}{(b+x^k)^{c+1}} \tag{51}$$

$$\leq (c_2 k c b^c / b^{c+1}) \int_0^\infty e^{-\alpha x} x^{k-1} dx$$

$$= c_2 k c \alpha^{-k} \Gamma(k)/b \quad ;$$

similarly

$$\delta(\infty,b,c) \leq k c \alpha^{-k} \Gamma(k)/b \quad . \tag{52}$$

Relations (50)-(52) imply (48). It can be easily checked that (48) implies (49). ∎

LEMMA 8. If $0 < \epsilon < \min(1, c_1)$ and

$$c_1 e^{-\alpha a} \left(\frac{b}{b+a^k}\right)^c \leq \epsilon/2 \tag{53}$$

$$c_2 k c b^c \int_a^\infty \frac{e^{-\alpha x} x^{k-1} dx}{(b+x^k)^{c+1}} \leq \epsilon/2 \quad , \tag{54}$$

then

$$|\phi(a,b,c) - \phi(\ ,b,c)| \leq \epsilon \tag{55}$$

$$|\tilde{\phi}(a,b,c) - \tilde{\phi}(\ ,b,c)| \leq \frac{\epsilon}{1-\delta(a,b,c)} \quad .$$

The straightforward proof is omitted.

Theorem 4 follows from the preceding Lemmas, algebraic manipulation, and Theorem 3.

7. Cost of Adapting

Suppose that recording data and varying replacement intervals entail a nonnegligible administrative cost with expected value $\psi(a,b,c)$ per stage until we stay with a fixed replacement interval. A new formulation is required.

Suppose that we will stop adapting on or before the N^{th} stage. Let $U_n(b,c)$ denote the minimal risk at stage n. Then

$$U_N(b,c) = R_{NN}(b,c) \tag{57}$$

and for $n < N$

$$U_n(b,c) = \min [U_{n+1}(b,c), U'_{n+1}(b,c)] \quad , \tag{58}$$

where

$$U'_n(b,c) = \min_a \{\psi(a,b,c) + \Phi(a,b,c) + T[a,U_n(b,c)]\} \quad . \tag{59}$$

The functions $U_n(b,c)$ are computed recursively for $n = N,\ldots,1$ over a finite grid of (b,c) values. For values off the grid, interpolation is used. If (b,c) is on the grid, then for $n = N,\ldots,1$ record

(i) $U_n(b,c)$.

(ii) whether the minimum was achieved for $U_{n+1}(b,c)$ or $U'_{n+1}(b,c)$.

(iii) in the latter case, the minimizing a .

To find the replacement interval to set a stage n', say, when the state is (b,c,t), find the smallest $n \geq n'$, say n^*, such that $U'_{n+1}(b,c) < U_{n+1}(b,c)$. If no such n is found, we use $a_{NN}(b,c)$.

Otherwise, we take the a that minimized U'^*_{n+1} as the replacement interval to set.

Note the resemblance to the structure of Bayes' procedures for truncated sequential games given in [2, Section 9.2]. $R_{NN}(b,c)$ plays the role of the terminal loss.

8. Finite Horizon

Suppose that we are given a finite time horizon H. We modify our loss function as follows.

$$L(a,x,t) = \begin{cases} c_1 e^{-\alpha(t+a)} & , \text{ if } x = a,\ t+a < H \\ c_2 e^{-\alpha(t+x)} & , \text{ if } x \neq a,\ t+x < H \\ 0 & , \text{ otherwise} \end{cases} \tag{60}$$

$$0 < c_1 < c_2 , \quad \alpha \geq 0 .$$

Since we now have a finite horizon, it is possible to let the discount rate vanish. It is clear that we may assume that $a \leq H-t$ and $t < H$.

If we stop at the end of the current stage, which we may suppose started at time t, then we incur a terminal loss $L_s(x,t)$. Our results do not depend on its form. Observe, however, that the optimal replacement interval to set will depend in general on t as well as on (b,c), in contrast to the infinite horizon case. We have the option of stopping immediately, making no further replacements of any kind; if we take this option, our terminal loss is $L_s(0,t)$. In general, we denote the expected value of $L_s(x,t)$ by $\Phi_s(a,b,c,t)$.

We assume that there is a finite upper bound N on the number of spares available. At each successive stage we select the optimal continuation. Then, à la [2, Section 9.2], the minimal risk with $N-n$ spares, $V_n(b,c,t)$, satisfies

$$V_N(b,c,t) = \min [L_s(0,t),\ Q(b,c,t)] , \tag{61}$$

where

$$(62) \qquad Q(b,c,t) = \min_{a} \left[\Phi_s(a,b,c,t) + e^{-\alpha t}\Phi(a,b,c) \right] \quad ,$$

and for $n < N$

$$(63) \qquad V_n(b,c,t) = \min \left[V_{n+1}(b,c,t),\ V'_{n+1}(b,c,t) \right] \quad ,$$

where

$$(64) \qquad V'_n(b,c,t) = \min_{a} \{ e^{-\alpha t}\Phi(a,b,c) + T'[a,V_n(b,c,t]] \}$$

and

$$(65) \qquad T'[a,V_n(b,c,t) = \frac{b}{b+a^k}^{c} V_n(b+a^k,c,t+a)$$

$$+ kcb^c \int_0^a V_n(b+x^k,c+1,t+x) \frac{x^{k-1}dx}{(b+x^k)^{c+1}} \quad .$$

The decisions

(i) whether or not to stop

(ii) if not, what replacement interval to set

are arrived at in a manner similar to that used in Chapter VII.

Reference to [1, Section 4.2.4] shows that, when the failure distribution is known, a much simpler formulation is possible.

Appendix I

Minimax Strategy When Discounting

Let the class of distributions $\mathcal{F}^{\alpha,\nu}$ be defined as follows: fix α,ν ($\alpha > 0$, $0 < \nu < 1$) ; a distribution F belongs to the class $\mathcal{F}^{\alpha,\nu}$ iff $\nu = \int_0^\infty e^{-\alpha x} dF(x)$. Let F_0 be the exponential distribution with mean $(1-\nu)/\alpha\nu$. It is easily checked that $F_0 \in \mathcal{F}^{\alpha,\nu}$.

THEOREM. With an infinite horizon and a loss function $L(a,x,t)$ given by (2) at each stage, the minimax strategy over $\mathcal{F}^{\alpha,\nu}$ is to replace only at failure. Replacement only at failure is also minimax over all distributions in $\mathcal{F}^{\alpha,\nu}$ having nondecreasing failure rate.

PROOF. For an arbitrary distribution F , denote by $R(a,F)$ the risk when a fixed replacement interval a is set at each stage. Using a technique found in [1, Section 4.2.1], we have

$$\frac{c_2\nu}{1-\nu} = \max_{F \in \mathcal{F}^{\alpha,\nu}} R(\infty,F) \geq \min_{a \geq 0} \max_{F \in \mathcal{F}^{\alpha,\nu}} R(a,F)$$

$$\geq \max_{F \in \mathcal{F}^{\alpha,\nu}} \min_{a \geq 0} R(a,F)$$

$$\geq \min_{a \geq 0} F(a,F_0) = R(\infty,F_0) = \frac{c_2\nu}{1-\nu} \quad . \qquad \|$$

Appendix II

Age Replacement When Discounting and the Failure Rate is Strictly Increasing to ∞

Let F have failure rate $q(\cdot)$, $0 < c_1 < c_2$, and $\alpha > 0$. Define

$$\bar{F}(a) = 1 - F(a)$$

$$\phi(a) = c_1 e^{-\alpha a}\bar{F}(a) + c_2 \int_0^a e^{-\alpha x} dF(x)$$

$$\delta(a) = e^{-\alpha x}\bar{F}(a) + \int_0^a e^{-\alpha x} dF(x) \quad .$$

$$R(a) = \frac{\phi(a)}{1-\delta(a)}$$

$$R(a^*) = \min_{a \geq 0} R(a) \quad .$$

THEOREM. If $q(a)$ is continuous and strictly increasing to ∞, then

(i) $a^* < \infty$

(ii) $q(a^*) = \alpha[c_1 + R(a^*)]/(c_2 - c_1)$.

(iii) a^* is unique .

PROOF.

(i) $$\frac{\partial}{\partial a} R(a) = \frac{e^{-\alpha a}\bar{F}(a)}{1-\delta(a)} \left[(c_2 - c_1)q(a) - \alpha(c_1 + R(a))\right] \quad ,$$

which is positive for all sufficiently large a, since $q(a) \uparrow \infty$ and $R(a)$ is uniformly bounded for $a \geq \epsilon > 0$.

(ii) By (i), a^* is a finite zero of $\frac{\partial}{\partial a} R(a)$. Such a zero exists since $q(a) \to \infty$, $R(a)$ is uniformly bounded for $a \geq \epsilon > 0$, $R(0) = \infty$, and $q(a)$ and $R(a)$ are continuous.

(iii) (suggested by David Matula).
Let a_1^* and a_2^* be minimizers. By (ii), $R(a_1^*) = R(a_2^*) \Longrightarrow q(a_1^*) = q(a_2^*) \Longrightarrow a_1^* = a_2^*$, since $q(a)$ is strictly increasing. ||

Note that $R(a)$ may have more than one local minimum, but that only one of these is a global minimum. If part (ii) of the theorem is used to find a^*, all local minima must be found and the corresponding risks evaluated. We suggest a possibly useful alternative procedure.

Let

$$V_{NN} = \min_a \Phi(a)$$

$$V_{nN} = \min_a \left[\Phi(a) + \delta(a) V_{n+1,N} \right], \quad n < N .$$

It can be shown that

$$q^{-1}(\alpha(c_1+V_{1N})/(c_2-c_1)) \uparrow a^*$$

by proving that $V_{1N} \uparrow R(a^*)$ (using the method of proof for Lemma 5) and comparison with part (ii) of the preceding theorem. The heuristic basis for this is that V_{1N} would be the minimal risk if we were to terminate after N stages. A monotone improving approximation to a^* is

$$q^{-1}(\alpha(c_1+V_{1N})/(c_2-c_1)) ,$$

where V_{1N} is computed by recursion. The minimizing a for V_{nN} is

$$q^{-1}(\alpha(c_1+V_{n+1,N})/(c_2-c_1)) ,$$

where we set $V_{N+1,N} = 0$. Note that q^{-1} exists since the failure rate is continuous and strictly increasing to ∞ . For the Weibull distribution defined by (1), $q^{-1}(x) = (x/\lambda k)^{1/(k-1)}$.

Appendix III

Interchange of lim and min

This appendix does not depend on the preceding material.

Let $\{f_n\}$ be a sequence of functions with values in the extended real line. We give sufficient conditions that

$$\lim_{n \to \infty} \min_{y \in Y} f_n(y) = \min_{y \in Y} \lim_{n \to \infty} f_n(y) \quad . \tag{66}$$

Although apparently not in the literature, the following result is probably well known and part of dynamic programming folklore:

LEMMA. If

(i) Y is compact

(ii) $f_n(y)$ is continuous in y, $\forall n$

(iii) $f_n \to f$, uniformly over Y,

then (66) holds.

PROOF. The first two conditions are included only to ensure that all minima in (66) exist, since a continuous function over a compact domain achieves its infimum; f is continuous, since it is the uniform limit of continuous functions.

Let $y^* \in Y$ be such that $f(y^*) = \min_{y \in Y} f(y)$ and $y_n^* \in Y$ be such that $f_n(y_n^*) = \min_{y \in Y} f_n(y)$, $n = 1,2,\ldots$. It suffices to show that

$$| f_n(y_n^*) - f(y^*) | = o(1) \quad . \tag{67}$$

Suppose that (67) does not hold. Then there exists $\epsilon > 0$ such that

$$|f_n(y_n^*) - f(y^*)| > \epsilon \quad \text{for infinitely many } n . \tag{68}$$

However, by (iii), there exists n' such that

$$|f_n(y) - f(y)| < \epsilon , \quad y \epsilon Y, \quad \forall n > n' . \tag{69}$$

Whether $f_n(y_n^*) \leq f(y^*)$ or vice versa, it follows that

$$\begin{aligned} |f_n(y_n^*) - f(y^*)| &\leq \max \{ |f_n(y_n^*) - f(y_n^*)| , \ |f_n(y^*) - f(y^*)| \} \\ &< \epsilon , \quad \forall n > n' , \end{aligned} \tag{70}$$

contradicting (68). ‖

We have not yet specified a topology for Y. In our case, Y was a compactified half line $[\epsilon, \infty]$ with the usual topology.

Appendix IV

Outline of Replacement Policy Computation

To simplify matters, we assume that the grid $\mathfrak{b}$ is given. See Chapter VI for further details. An outline of the replacement policy computation is given by the following algorithm:

1. Compute $R_{11}(b,c)$ for all $(b,c)\epsilon\mathfrak{b}$.

2. $N = 2$.

3. Compute $R_{1N}(b,c)$ for all $(b,c)\epsilon\mathfrak{b}$ using the previously computed values of $R_{1,N-1}(b,c)$. [Note that $R_{1,N-1}(b,c) = R_{2N}(b,c)$] .

4. If $\max_{(b,c)\epsilon\mathfrak{b}} \left| \frac{R_{1,N-1}(b,c)-R_{1N}(b,c)}{R_{1,N-1}(b,c)} \right| < \epsilon$,

   go to 7; otherwise go to 5 .

5. $N \rightarrow N+1$.

6. Go to 3 .

7. Exit with a table of $R_{1N}(b,c)$ and $a_{1N}(b,c)$ values.

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